CHAPTER 6
Counting

SECTION 6.1 The Basics of Counting

2. By the product rule there are $27 \cdot 37 = 999$ offices.

4. By the product rule there are $12 \cdot 2 \cdot 3 = 72$ different types of shirt.

6. By the product rule there are $4 \cdot 6 = 24$ routes.

8. There are 26 choices for the first initial, then 25 choices for the second, if no letter is to be repeated, then 24 choices for the third. (We interpret "repeated" broadly, so that a string like RWR, for example, is prohibited, as well as a string like RRW.) Therefore by the product rule the answer is $26 \cdot 25 \cdot 24 = 15{,}600$.

10. We have two choices for each bit, so there are $2^8 = 256$ bit strings.

12. We use the sum rule, adding the number of bit strings of each length up to 6. If we include the empty string, then we get $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$ (using the formula for the sum of a geometric progression—see Theorem 1 in Section 2.4).

14. If $n = 0$, then the empty string—vacuously—satisfies the condition (or does not, depending on how one views it). If $n = 1$, then there is one, namely the string 1. If $n \geq 2$, then such a string is determined by specifying the $n - 2$ bits between the first bit and the last, so there are 2^{n-2} such strings.

16. We can subtract from the number of strings of length 4 of lower case letters the number of strings of length 4 of lower case letters other than x. Thus the answer is $26^4 - 25^4 = 66{,}351$.

18. Recall that a DNA sequence is a sequence of letters, each of which is one of A, C, G, or T. Thus by the product rule there are $4^5 = 1024$ DNA sequences of length five if we impose no restrictions.

a) If the sequence must end with A, then there are only four positions at which to make a choice, so the answer is $4^4 = 256$.

b) If the sequence must start with T and end with G, then there are only three positions at which to make a choice, so the answer is $4^3 = 64$.

c) If only two letters can be used rather than four, the number of choices is $2^5 = 32$.

d) As in part **(c)**, there are $3^5 = 243$ sequences that do not contain C.

20. Because neither 5 nor 31 is divisible by either 3 or 4, whether the ranges are meant to be inclusive or exclusive of their endpoints is moot.

a) There are $\lfloor 31/3 \rfloor = 10$ integers less than 31 that are divisible by 3, and $\lfloor 5/3 \rfloor = 1$ of them is less than 5 as well. This leaves $10 - 1 = 9$ numbers between 5 and 31 that are divisible by 3. They are 6, 9, 12, 15, 18, 21, 24, 27, and 30.

b) There are $\lfloor 31/4 \rfloor = 7$ integers less than 31 that are divisible by 4, and $\lfloor 5/4 \rfloor = 1$ of them is less than 5 as well. This leaves $7 - 1 = 6$ numbers between 5 and 31 that are divisible by 4. They are 8, 12, 16, 20, 24, and 28.

c) A number is divisible by both 3 and 4 if and only if it is divisible by their least common multiple, which is 12. Obviously there are two such numbers between 5 and 31, namely 12 and 24. We could also work this out as we did in the previous parts: $\lfloor 31/12 \rfloor - \lfloor 5/12 \rfloor = 2 - 0 = 2$. Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

22. **a)** Every seventh number is divisible by 7. Therefore there are $\lfloor 999/7 \rfloor = 142$ such numbers. Note that we use the floor function, because the k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) For solving this part and the next four parts, we need to use the principle of inclusion–exclusion. Just as in part **(a)**, there are $\lfloor 999/11 \rfloor = 90$ numbers in our range divisible by 11, and there are $\lfloor 999/77 \rfloor = 12$ numbers in our range divisible by both 7 and 11 (the multiples of 77 are the numbers we seek). If we take these 12 numbers away from the 142 numbers divisible by 7, we see that there are 130 numbers in our range divisible by 7 but not 11.

c) As explained in part **(b)**, the answer is 12.

d) By the principle of inclusion–exclusion, the answer, using the data from part **(b)**, is $142 + 90 - 12 = 220$.

e) If we subtract from the answer to part **(d)** the number of numbers divisible by both 7 and 11, we will have the number of numbers divisible by neither of them; so the answer is $220 - 12 = 208$.

f) If we subtract the answer to part **(d)** from the total number of positive integers less than 1000, we will have the number of numbers divisible by exactly one of them; so the answer is $999 - 220 = 779$.

g) If we assume that numbers are written without leading 0's, then we should break the problem down into three cases—one-digit numbers, two-digit numbers and three-digit numbers. Clearly there are 9 one-digit numbers, and each of them has distinct digits. There are 90 two-digit numbers (10 through 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. An alternative way to compute this is to note that the first digit must be 1 through 9 (9 choices), and the second digit must be something different from the first digit (9 choices out of the 10 possible digits), so by the product rule, we get $9 \cdot 9 = 81$ choices in all. This approach also tells us that there are $9 \cdot 9 \cdot 8 = 648$ three-digit numbers with distinct digits (again, work from left to right—in the ones place, only 8 digits are left to choose from). So the final answer is $9 + 81 + 648 = 738$.

h) It turns out to be easier to count the odd numbers with distinct digits and subtract from our answer to part **(g)**, so let us proceed that way. There are 5 odd one-digit numbers. For two-digit numbers, first choose the ones digit (5 choices), then choose the tens digit (8 choices), since neither the ones digit value nor 0 is available); therefore there are 40 such two-digit numbers. (Note that this is not exactly half of 81.) For the three-digit numbers, first choose the ones digit (5 choices), then the hundreds digit (8 choices), then the tens digit (8 choices, giving us 320 in all. So there are $5 + 40 + 320 = 365$ odd numbers with distinct digits. Thus the final answer is $738 - 365 = 373$.

24. It will be useful to note first that there are exactly 9000 numbers in this range.

a) Every ninth number is divisible by 9, so the answer is one ninth of 9000 or 1000.

b) Every other number is even, so the answer is one half of 9000 or 4500.

c) We can reason from left to right. There are 9 choices for the first (left-most) digit (since it cannot be a 0), then 9 choices for the second digit (since it cannot equal the first digit), then, in a similar way, 8 choices for the third digit, and 7 choices for the right-most digit. Therefore there are $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ ways to specify such a number. In other words, there are 4536 such numbers. Note that this coincidentally turns out to be almost exactly half of the numbers in the range.

d) Every third number is divisible by 3, so one third of 9000 or 3000 numbers in this range are divisible by 3. The remaining 6000 are not.

e) For this and the next three parts we need to note first that one fifth of the numbers in this range, or 1800 of them, are divisible by 5, and one seventh of them, or 1286 are divisible by 7. [This last calculation is a little more subtle than we let on, since 9000 is not divisible by 7 (the quotient is 1285.71...). But 1001 is divisible by 7, and $1001 + 1285 \cdot 7 = 9996$, so there are indeed 1286, and not 1285 such multiples. (By contrast, in the range 1002 to 10001, inclusive, which also includes 9000 numbers, there are only 1285 multiples of 7.)] We also need to know how many of these numbers are divisible by both 5 and 7, which means divisible by 35. The answer, by the similar reasoning, is 257, namely those multiples from $29 \cdot 35 = 1015$ to $285 \cdot 35 = 9975$. (One more note: We could also have come up with these numbers more formally, using the ideas in Section 8.5, especially Example 2. We could find the number of multiples less than 10,000 and subtract the number of multiples less than 1000.) Now to the problem at hand. The number of numbers divisible by 5 or 7 is the number of numbers divisible by 5, plus the number of numbers divisible by 7, minus (because of having overcounted) the number of numbers divisible by both. So our answer is $1800 + 1286 - 257 = 2829$.

f) Since we just found that 2829 of these numbers are divisible by either 5 or 7, it follows that the rest of them, $9000 - 2829 = 6171$, are not.

g) We noted in the solution to part **(e)** that 1800 numbers are divisible by 5, and 257 of these are also divisible by 7. Therefore $1800 - 257 = 1543$ numbers in our range are divisible by 5 but not by 7.

h) We found this as part of our solution to part **(e)**, namely 257.

26. **a)** There are 10 ways to choose the first digit, 9 ways to choose the second, and so on; therefore the answer is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

b) There are 10 ways to choose each of the first three digits and 5 ways to choose the last; therefore the answer is $10^3 \cdot 5 = 5000$.

c) There are 4 ways to choose the position that is to be different from 9, and 9 ways to choose the digit to go there. Therefore there are $4 \cdot 9 = 36$ such strings.

28. $10^3 26^3 + 26^3 10^3 = 35{,}152{,}000$

30. $26^3 10^3 + 26^4 10^2 = 63{,}273{,}600$

32. **a)** By the product rule, the answer is $26^8 = 208{,}827{,}064{,}576$.

b) By the product rule, the answer is $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 62{,}990{,}928{,}000$.

c) This is the same as part **(a)**, except that there are only seven slots to fill, so the answer is $26^7 = 8{,}031{,}810{,}176$.

d) This is similar to **(b)**, except that there is only one choice in the first slot, rather than 26, so the answer is $1 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 2{,}422{,}728{,}000$.

e) This is the same as part **(c)**, except that there are only six slots to fill, so the answer is $26^6 = 308{,}915{,}776$.

f) This is the same as part **(e)**; again there are six slots to fill, so the answer is $26^6 = 308{,}915{,}776$.

g) This is the same as part **(f)**, except that there are only four slots to fill, so the answer is $26^4 = 456{,}976$. We are assuming that the question means that the legal strings are BO????BO, where any letters can fill the middle four slots.

h) By part **(f)**, there are 26^6 strings that start with the letters BO in that order. By the same argument, there are 26^6 strings that end that way. By part **(g)**, there are 26^4 strings that both start and end with the letters BO in that order. Therefore by the inclusion–exclusion principle, the answer is $26^6 + 26^6 - 26^4 = 617{,}374{,}576$.

34. In each case the answer is n^{10}, where n is the number of elements in the codomain, since there are n choices for a function value for each of the 10 elements in the domain.

a) $2^{10} = 1024$ **b)** $3^{10} = 59{,}049$ **c)** $4^{10} = 1{,}048{,}576$ **d)** $5^{10} = 9{,}765{,}625$

36. There are 2^n such functions, since there is a choice of 2 function values for each element of the domain.

38. By our solution to Exercise 39, the answer is $(n+1)^5$ in each case, where n is the number of elements in the codomain.
a) $2^5 = 32$ **b)** $3^5 = 243$ **c)** $6^5 = 7776$ **d)** $10^5 = 100{,}000$

40. We know that there are 2^{100} subsets in all. Clearly 101 of them do not have more than one element, namely the empty set and the 100 sets consisting of 1 element. Therefore the answer is $2^{100} - 101 \approx 1.3 \times 10^{30}$.

42. Recall that a DNA sequence is a sequence of letters, each of which is one of A, C, G, or T. Thus by the product rule there are $4^4 = 256$ DNA sequences of length four if we impose no restrictions.
a) If the letter T cannot be used, then the number of choices is $3^4 = 81$.
b) The sequence must be either ACGx or xACG, where x is one of the four letters. These two cases do not overlap, so the answer is $4 + 4 = 8$.
c) There are four positions and four letters, each used exactly once. There are 4 choices for the first position, then 3 for the second, 2 for the third, and 1 for the fourth. Therefore the answer is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.
d) There are four ways to choose which letter is to be occur twice and three ways to decide which of the other letters to leave out, so there are $4 \cdot 3 = 12$ choices of the letters for the sequence. There are 4 positions the first (alphabetically) of the single-use letters can occupy, and then 3 positions for the second single-use letter, a total of $4 \cdot 3 = 12$ different sequences once we have determined the letters and their frequencies. Therefore the answer is $12 \cdot 12 = 144$.

44. If we ignore the fact that the table is round and just count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements. However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so this overcounts by a factor of 4. (For example, the sequence Mary–Debra–Cristina–Julie gives the same circular seating as the sequence Julie–Mary–Debra–Cristina.) Therefore the answer is $5040/4 = 1260$.

46. **a)** We first place the bride in any of the 6 positions. Then, from left to right in the remaining positions, we choose the other five people to be in the picture; this can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ ways. Therefore the answer is $6 \cdot 15120 = 90{,}720$.
b) We first place the bride in any of the 6 positions, and then place the groom in any of the 5 remaining positions. Then, from left to right in the remaining positions, we choose the other four people to be in the picture; this can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways. Therefore the answer is $6 \cdot 5 \cdot 1680 = 50{,}400$.
c) From part **(a)** there are 90720 ways for the bride to be in the picture. There are (from part **(b)**) 50400 ways for both the bride and groom to be in the picture. Therefore there are $90720 - 50400 = 40320$ ways for just the bride to be in the picture. Symmetrically, there are 40320 ways for just the groom to be in the picture. Therefore the answer is $40320 + 40320 = 80{,}640$.

48. There are 2^5 strings that begin with two 0's (since there are two choices for each of the last five bits). Similarly there are 2^4 strings that end with three 1's. Furthermore, there are 2^2 strings that both begin with two 0's and end with three 1's (since only bits 3 and 4 are free to be chosen). By the inclusion–exclusion principle, there are $2^5 + 2^4 - 2^2 = 44$ such strings in all.

50. First we count the number of bit strings of length 10 that contain five consecutive 0's. We will base the count on where the string of five or more consecutive 0's starts. If it starts in the first bit, then the first five bits are all 0's, but there is free choice for the last five bits; therefore there are $2^5 = 32$ such strings. If it starts in the second bit, then the first bit must be a 1, the next five bits are all 0's, but there is free choice for the last four bits; therefore there are $2^4 = 16$ such strings. If it starts in the third bit, then the second bit must be a 1 but the first bit and the last three bits are arbitrary; therefore there are $2^4 = 16$ such strings. Similarly, there are 16 such strings that have the consecutive 0's starting in each of positions four, five, and six. This gives us a total of $32 + 5 \cdot 16 = 112$ strings that contain five consecutive 0's. Symmetrically, there are 112 strings that contain five consecutive 1's. Clearly there are exactly two strings that contain both (0000011111 and 1111100000). Therefore by the inclusion–exclusion principle, the answer is $112 + 112 - 2 = 222$.

52. This is a straightforward application of the inclusion–exclusion principle: $38+23-7 = 54$ (we need to subtract the 7 double majors counted twice in the sum).

54. Order matters here, since the initials RSZ, for example, are different from the initials SRZ. By the sum rule we can add the number of initials formable with two, three, four, and five letters. By the product rule, these are 26^2, 26^3, 26^4, and 26^5, respectively, so the answer is $676 + 17576 + 456976 + 11881376 = 12{,}356{,}604$.

56. We need to compute the number of variable names of length i for $i = 1, 2, \ldots, 8$, and add. A variable name of length i is specified by choosing a first character, which can be done in 53 ways ($2 \cdot 26$ letters and 1 underscore to choose from), and $i - 1$ other characters, each of which can be done in $53 + 10 = 63$ ways. Therefore the answer is

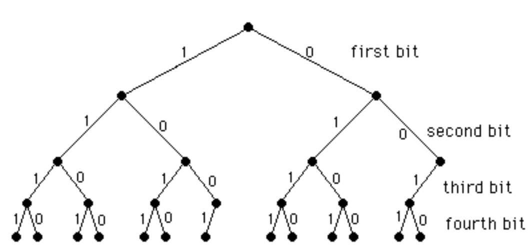
$$\sum_{i=1}^{8} 52 \cdot 63^{i-1} = 52 \cdot \frac{63^8 - 1}{63 - 1} \approx 2.1 \times 10^{14}\,.$$

58. There are $10 - 1 = 9$ country codes of length 1, $10^2 = 100$ of length 2, and $10^3 = 1000$ of length 3, for a total of 1109 country codes. The number of numbers following the country code is $10 + 10^2 + 10^3 + \cdots + 10^{15}$; by the formula for a geometric series (Theorem 1 in Section 2.4), this equals $10(10^{15} - 1)/(10 - 1) = 1{,}111{,}111{,}111{,}111{,}110$. Therefore there are $1109 \cdot 1{,}111{,}111{,}111{,}111{,}110 = 1{,}232{,}222{,}222{,}222{,}220{,}990$ possible numbers.

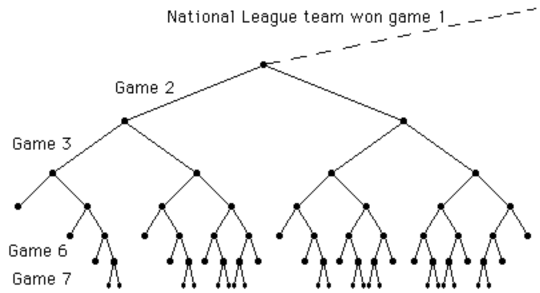
60. By the sum and product rules, the answer is $26^3 + 26^4 + 26^5 + 26^6 = 321{,}271{,}704$.

62. Let P be the set of numbers in $\{1, 2, 3, \ldots, n\}$ that are divisible by p, and similarly define the set Q. We want to count the numbers not divisible by either p or q, so we want $n - |P \cup Q|$. By the principle of inclusion–exclusion, $|P \cup Q| = |P| + |Q| - |P \cap Q|$. Every p^{th} number is divisible by p, so $|P| = \lfloor n/p \rfloor$. Similarly $|Q| = \lfloor n/q \rfloor$. Clearly n is the only positive integer not exceeding n that is divisible by both p and q, so $|P \cap Q| = 1$. Therefore the number of positive integers not exceeding n that are relatively prime to n is $n - (\lfloor n/p \rfloor + \lfloor n/q \rfloor - 1) = n - \lfloor n/p \rfloor - \lfloor n/q \rfloor + 1$.

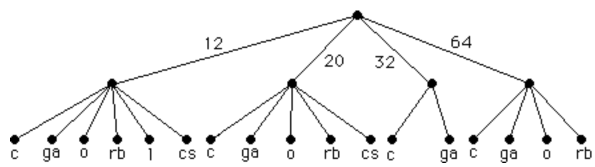
64. We draw the tree, with its root at the top. We show a branch for each of the possibilities 0 and 1, for each bit in order, except that we do not allow three consecutive 0's. Since there are 13 leaves, the answer is 13.



66. The tree is a bit too large to draw in its entirety. We show only half of it, namely the half corresponding to the National League team's having won the first game. By symmetry, the final answer will be twice the number computed with this tree. A branch to the left indicates a win by the National League team; a branch to the right, a win by the American league team. No further branching occurs whenever one team has won four games. Since we see 35 leaves, the answer is 70.



68. a) It is more convenient to branch on bottle size first. Note that there are a different number of branches coming off each of the nodes at the second level. The number of leaves in the tree is 17, which is the answer.



b) We can add the number of different varieties for each of the sizes. The 12-ounce bottle has 6, the 20-ounce bottle has 5, the 32-once bottle has 2, and the 64-ounce bottle has 4. Therefore $6+5+2+4=17$ different types of bottles need to be stocked.

70. There are 2^n lines in the truth table, since each of the n propositions can have 2 truth values. Each line can be filled in with T or F, so there are a total of 2^{2^n} possibilities.

72. We want to show that a procedure consisting of m tasks can be done in $n_1n_2\cdots n_m$ ways, if the i^{th} task can be done in n_i ways. The product rule stated in the text is the basis step, $m=2$. Assume the inductive hypothesis. Then to do the procedure we have to do each of the first m tasks, which by the inductive hypothesis can be done in $n_1n_2\cdots n_m$ ways, and then the $(m+1)^{\text{st}}$ task, so there are $(n_1n_2\cdots n_m)n_{m+1}$ possibilities, as desired.

74. a) The largest value of TOTAL LENGTH is $2^{16}-1$, since this would be the number represented by a string of 16 1's. So the maximum length of a datagram is 65,535 octets (or bytes).

b) The largest value of HLEN is $2^4-1=15$, since this would be the number represented by a string of four 1's. So the maximum length of a header is 15 32-bit blocks. Since there are four 8-bit octets (or bytes) in a block, the maximum length of the header is $4\cdot 15=60$ octets.

c) We saw in part **(a)** that the maximum total length is 65,535 octets. If at least 20 of these must be devoted to the header, the data area can be at most 65,515 octets long.

d) There are $2^8=256$ different octets, since each bit of an octet can be 0 or 1. In part **(c)** we saw that the data area could be at most 65,515 octets long. So the answer is 256^{65515}, which is a huge number (approximately 7×10^{157775}, according to a computer algebra system).

SECTION 6.2 The Pigeonhole Principle

2. This follows from the pigeonhole principle, with $k = 26$.

4. We assume that the woman does not replace the balls after drawing them.

a) There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains three pigeons. By the generalized pigeonhole principle, the answer is 5. If five balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color. On the other hand four balls is not enough, because two might be red and two might be blue. Note that the number of balls was irrelevant (assuming that it was at least 5).

b) She needs to select 13 balls in order to insure at least three blue ones. If she does so, then at most 10 of them are red, so at least three are blue. On the other hand, if she selects 12 or fewer balls, then 10 of them could be red, and she might not get her three blue balls. This time the number of balls did matter.

6. There are only d possible remainders when an integer is divided by d, namely $0, 1, \ldots, d-1$. By the pigeonhole principle, if we have $d+1$ remainders, then at least two must be the same.

8. This is just a restatement of the pigeonhole principle, with $k = |T|$.

10. The midpoint of the segment whose endpoints are (a,b) and (c,d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even) and b and d have the same parity. Thus what matters in this problem is the parities of the coordinates. There are four possible pairs of parities: (odd, odd), $(\text{odd}, \text{even})$, $(\text{even}, \text{odd})$, and $(\text{even}, \text{even})$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

12. This is similar in spirit to Exercise 10. Working modulo 5 there are 25 pairs: $(0,0)$, $(0,1)$, $\ldots$, $(4,4)$. Thus we could have 25 ordered pairs of integers (a,b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

14. **a)** We can group the first ten positive integers into five subsets of two integers each, each subset adding up to 11: $\{1,10\}$, $\{2,9\}$, $\{3,8\}$, $\{4,7\}$, and $\{5,6\}$. If we select seven integers from this set, then by the pigeonhole principle at least two of them come from the same subset. Furthermore, if we forget about these two in the same group, then there are five more integers and four groups; again the pigeonhole principle guarantees two integers in the same group. This gives us two pairs of integers, each pair from the same group. In each case these two integers have a sum of 11, as desired.

b) No. The set $\{1,2,3,4,5,6\}$ has only 5 and 6 from the same group, so the only pair with sum 11 is 5 and 6.

16. We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 16, namely $\{1,15\}$, $\{3,13\}$, $\{5,11\}$, and $\{7,9\}$. If we select five numbers from the set $\{1,3,5,7,9,11,13,15\}$, then at least two of them must fall within the same subset, since there are only four subsets. Two numbers in the same subset are the desired pair that add up to 16. We also need to point out that choosing four numbers is not enough, since we could choose $\{1,3,5,7\}$, and no pair of them add up to more than 12.

18. **a)** If not, then there would be 4 or fewer male students and 4 or fewer female students, so there would be $4+4=8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

b) If not, then there would be 2 or fewer male students and 6 or fewer female students, so there would be $2+6=8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

20. One maximal length increasing sequence is $5, 7, 10, 15, 21$. One maximal length decreasing sequence is $22, 7, 3$. See Exercise 25 for an algorithm.

22. This follows immediately from Theorem 3, with $n = 10$.

24. This problem was on the International Mathematical Olympiad in 2001, a test taken by the six best high school students from each country. Here is a paraphrase of a solution posted on the Web by Steve Olson, author of a book about this competition entitled *Count Down*. Make a table listing the 21 boys at the top of each column and the 21 girls to the left of each row. This table will contain $21 \cdot 21 = 441$ boxes. In each box write the number of a problem solved by both that girl and that boy. From the given information, each box will contain a number. Each contestant solved at most six problems, so only six different numbers can appear in any given row or column of 21 boxes. Because $5 \cdot 2 = 10$, at least $21 - 10 = 11$ of the boxes in any given row or column must contain problem numbers that appear three or more times in that row. (This is an application of the idea of the pigeonhole principle.) In each row color red all the boxes containing problem numbers that appear at least three times in that row. So each row will have at least 11 red boxes, and therefore there will be at least $11 \cdot 21 = 231$ boxes colored red. Repeat the process with the columns, using the color blue. Because at least 231 boxes are red and 231 are blue, and there are only 441 boxes in all, some of the boxes will be both red and blue. (Here is the second place where the pigeonhole principle is used.) The problem number in a doubly-colored box represents a problem solved by at least three girls and at least three boys.

26. Let the people be A, B, C, D, and E. Suppose the following pairs are friends: $A–B$, $B–C$, $C–D$, $D–E$, and $E–A$. The other five pairs are enemies. In this example, there are no three mutual friends and no three mutual enemies.

28. Let A be one of the people. She must have either 10 friends or 10 enemies, since if there were 9 or fewer of each, then that would account for at most 18 of the 19 other people. Without loss of generality assume that A has 10 friends. By Exercise 27 there are either 4 mutual enemies among these 10 people, or 3 mutual friends. In the former case we have our desired set of 4 mutual enemies; in the latter case, these 3 people together with A form the desired set of 4 mutual friends.

30. This is clear by symmetry, since we can just interchange the notions of friends and enemies.

32. There are 99,999,999 possible positive salaries less than one million dollars, i.e., from \$0.01 to \$999,999.99. By the pigeonhole principle, if there were more than this many people with positive salaries less than one million dollars, then at least two of them must have the same salary.

34. This follows immediately from Theorem 2, with $N = 8{,}008{,}278$ and $k = 1{,}000{,}001$ (the number of hairs can be anywhere from 0 to a million).

36. Let $K(x)$ be the number of other computers that computer x is connected to. The possible values for $K(x)$ are $1, 2, 3, 4, 5$. Since there are 6 computers, the pigeonhole principle guarantees that at least two of the values $K(x)$ are the same, which is what we wanted to prove.

38. This is similar to Example 9. Label the computers C_1 through C_8, and label the printers P_1 through P_4. If we connect C_k to P_k for $k = 1, 2, 3, 4$ and connect each of the computers C_5 through C_8 to *all* the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables. Clearly this is sufficient, because if computers C_1 through C_4 need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 19 cables are not enough. Since there are 19 cables and 4 printers, the average number of computers per printer is $19/4$, which is less than 5. Therefore some printer must be connected to fewer than 5 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 4 or fewer computers, so there are at least 4 computers that are not connected to it. If those 4 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 3 other printers.

40. Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

42. **a)** The solution of Exercise 41, with 24 replaced by 2 and 149 replaced by 127, tells us that the statement is true.

b) The solution of Exercise 41, with 24 replaced by 23 and 149 replaced by 148, tells us that the statement is true.

c) We begin in a manner similar to the solution of Exercise 41. Look at a_1, a_2, ..., a_{75}, a_1+25, ..., $a_{75}+25$, where a_i is the total number of matches played up through and including hour i. Then $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$, and $26 \leq a_1 + 25 < a_2 + 25 < \cdots < a_{75} + 25 \leq 150$. Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, as in Exercise 41, $a_i = a_j + 25$ for some i and j and we are done. In the former case, however, since each of the numbers $a_i + 25$ is greater than or equal to 26, the numbers $1, 2, \ldots, 25$ must all appear among the a_i's. But since the a_i's are increasing, the only way this can happen is if $a_1 = 1$, $a_2 = 2$, ..., $a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

d) We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let a_1, a_2, ..., a_{75} be as before, and note that $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. By the pigeonhole principle two of the numbers among a_1, a_2, ..., a_{31} are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 60 or more, so $a_{31} \geq 61$. Similarly, among a_{31} through a_{61}, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

44. Look at the pigeonholes $\{1000, 1001\}$, $\{1002, 1003\}$, $\{1004, 1005\}$, ..., $\{1098, 1099\}$. There are clearly 50 sets in this list. By the pigeonhole principle, if we have 51 numbers in the range from 1000 to 1099 inclusive, then at least two of them must come from the same set. These are the desired two consecutive house numbers.

46. Suppose this statement were not true. Then for each i, the i^{th} box contains at most $n_i - 1$ objects. Adding, we have at most $(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n_1 + n_2 + \cdots + n_t - t$ objects in all, contradicting the fact that there were $n_1 + n_2 + \cdots + n_t - t + 1$ objects in all. Therefore the statement must be true.

SECTION 6.3 Permutations and Combinations

2. $P(7,7) = 7! = 5040$

4. There are 10 combinations and 60 permutations. We list them in the following way. Each combination is listed, without punctuation, in increasing order, followed by the five other permutations involving the same numbers, in parentheses, without punctuation.

123 (132 213 231 312 321) 124 (142 214 241 412 421) 125 (152 215 251 512 521)

134 (143 314 341 413 431) 135 (153 315 351 513 531) 145 (154 415 451 514 541)

234 (243 324 342 423 432) 235 (253 325 352 523 532)

245 (254 425 452 524 542) 345 (354 435 453 534 543)

6. **a)** $C(5,1) = 5$ **b)** $C(5,3) = C(5,2) = 5 \cdot 4/2 = 10$ **c)** $C(8,4) = 8 \cdot 7 \cdot 6 \cdot 5/(4 \cdot 3 \cdot 2) = 70$
d) $C(8,8) = 1$ **e)** $C(8,0) = 1$ **f)** $C(12,6) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7/(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 924$

8. $P(5,5) = 5! = 120$

10. $P(6,6) = 6! = 720$

12. **a)** To specify a bit string of length 12 that contains exactly three 1's, we simply need to choose the three positions that contain the 1's. There are $C(12,3) = 220$ ways to do that.

b) To contain at most three 1's means to contain three 1's, two 1's, one 1, or no 1's. Reasoning as in part **(a)**, we see that there are $C(12,3)+C(12,2)+C(12,1)+C(12,0) = 220+66+12+1 = 299$ such strings.

c) To contain at least three 1's means to contain three 1's, four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, 10 1's, 11 1's, or 12 1's. We could reason as in part **(b)**, but we would have too many numbers to add. A simpler approach would be to figure out the number of ways not to have at least three 1's (i.e., to have two 1's, one 1, or no 1's) and then subtract that from 2^{12}, the total number of bit strings of length 12. This way we get $4096 - (66 + 12 + 1) = 4017$.

d) To have an equal number of 0's and 1's in this case means to have six 1's. Therefore the answer is $C(12,6) = 924$.

14. $C(99,2) = 99 \cdot 98/2 = 4851$

16. We need to compute $C(10,1) + C(10,3) + C(10,5) + C(10,7) + C(10,9) = 10 + 120 + 252 + 120 + 10 = 512$. (In the next section we will see that there are just as many subsets with an odd number of elements as there are subsets with an even number of elements (Exercise 31 in Section 6.4). Since there are $2^{10} = 1024$ subsets in all, the answer is $1024/2 = 512$, in agreement with our computation.)

18. **a)** Each flip can be either heads or tails, so there are $2^8 = 256$ possible outcomes.

b) To specify an outcome that has exactly three heads, we simply need to choose the three flips that came up heads. There are $C(8,3) = 56$ such outcomes.

c) To contain at least three heads means to contain three heads, four heads, five heads, six heads, seven heads, or eight heads. Reasoning as in part **(b)**, we see that there are $C(8,3)+C(8,4)+C(8,5)+C(8,6)+C(8,7)+C(8,8) = 56 + 70 + 56 + 28 + 8 + 1 = 219$ such outcomes. We could also subtract from 256 the number of ways to get two or fewer heads, namely $28 + 8 + 1 = 37$. Since $256 - 37 = 219$, we obtain the same answer using this alternative method.

d) To have an equal number of heads and tails in this case means to have four heads. Therefore the answer is $C(8,4) = 70$.

20. **a)** There are $C(10,3)$ ways to choose the positions for the 0's, and that is the only choice to be made, so the answer is $C(10,3)=120$.

b) There are more 0's than 1's if there are fewer than five 1's. Using the same reasoning as in part **(a)**, together with the sum rule, we obtain the answer $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4) = 1+10+45+120+210=386$. Alternatively, by symmetry, half of all cases in which there are not five 0's have more 0's than 1's; therefore the answer is $(2^{10}-C(10,5)/2=(1024-252)/2=386$.

c) We want the number of bit strings with 7, 8, 9, or 10 1's. By the same reasoning as above, there are $C(10,7)+C(10,8)+C(10,9)+C(10,10)=120+45+10+1=176$ such strings.

d) If a string does not have at least three 1's, then it has 0, 1, or 2 1's. There are $C(10,0)+C(10,1)+C(10,2)=1+10+45=56$ such strings. There are $2^{10}=1024$ strings in all. Therefore there are $1024-56=968$ strings with at least three 1's.

22. **a)** If ED is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of seven items—the letters A, B, C, F, G, and H, and the superletter ED. Therefore the answer is $P(7,7)=7!=5040$.

b) Reasoning as in part **(a)**, we see that the answer is $P(6,6)=6!=720$.

c) As in part **(a)**, we glue BA into one item and glue FGH into one item. Therefore we need to permute five items, and there are $P(5,5)=5!=120$ ways to do it.

d) This is similar to part **(c)**. Glue AB into one item, glue DE into one item, and glue GH into one item, producing five items, so the answer is $P(5,5)=5!=120$.

e) If both CAB and BED are substrings, then $CABED$ has to be a substring. So we are really just permuting four items: $CABED$, F, G, and H. Therefore the answer is $P(4,4)=4!=24$.

f) There are no permutations with both of these substrings, since B cannot be followed by both C and F at the same time.

24. First position the women relative to each other. Since there are 10 women, there are $P(10,10)$ ways to do this. This creates 11 slots where a man (but not more than one man) may stand: in front of the first woman, between the first and second women, ..., between the ninth and tenth women, and behind the tenth woman. We need to choose six of these positions, in order, for the first through six man to occupy (order matters, because the men are distinct people). This can be done is $P(11,6)$ ways. Therefore the answer is $P(10,10)\cdot P(11,6)=10!\cdot 11!/5!=1{,}207{,}084{,}032{,}000$.

26. **a)** This is just a matter of choosing 10 players from the group of 13, since we are not told to worry about what positions they play; therefore the answer is $C(13,10)=286$.

b) This is the same as part **(a)**, except that we need to worry about the order in which the choices are made, since there are 10 distinct positions to be filled. Therefore the answer is $P(13,10)=13!/3!=1{,}037{,}836{,}800$.

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men. Therefore (using part **(a)**) there are $286-1=285$ ways to choose the players if at least one of them must be a woman.

28. We are just being asked for the number of strings of T's and F's of length 40 with exactly 17 T's. The only choice is which 17 of the 40 positions are to have the T's, so the answer is $C(40,17)\approx 8.9\times 10^{10}$.

30. **a)** There are $C(16,5)$ ways to select a committee if there are no restrictions. There are $C(9,5)$ ways to select a committee from just the 9 men. Therefore there are $C(16,5)-C(9,5)=4368-126=4242$ committees with at least one woman.

b) There are $C(16,5)$ ways to select a committee if there are no restrictions. There are $C(9,5)$ ways to select a committee from just the 9 men. There are $C(7,5)$ ways to select a committee from just the 7 men. These two possibilities do not overlap, since there are no ways to select a committee containing neither men nor women. Therefore there are $C(16,5) - C(9,5) - C(7,5) = 4368 - 126 - 21 = 4221$ committees with at least one woman and at least one man.

32. **a)** The only reasonable way to do this is by subtracting from the number of strings with no restrictions the number of strings that do not contain the letter a. The answer is $26^6 - 25^6 = 308915776 - 244140625 = 64{,}775{,}151$.

b) If our string is to contain both of these letters, then we need to subtract from the total number of strings the number that fail to contain one or the other (or both) of these letters. As in part **(a)**, 25^6 strings fail to contain an a; similarly 25^6 fail to contain a b. This is overcounting, however, since 24^6 fail to contain both of these letters. Therefore there are $25^6 + 25^6 - 24^6$ strings that fail to contain at least one of these letters. Therefore the answer is $26^6 - (25^6 + 25^6 - 24^6) = 308915776 - (244140625 + 244140625 - 191102976) = 11{,}737{,}502$.

c) First choose the position for the a; this can be done in 5 ways, since the b must follow it. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $5P(24,4) = 1{,}275{,}120$.

d) First choose the positions for the a and b; this can be done in $C(6,2)$ ways, since once we pick two positions, we put the a in the left-most and the b in the other. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $C(6,2)P(24,4) = 3{,}825{,}360$.

34. Probably the best way to do this is just to break it down into the three cases by sex. There are $C(15,6)$ ways to choose the committee to be composed only of women, $C(15,5)C(10,1)$ ways if there are to be five women and one man, and $C(15,4)C(10,2)$ ways if there are to be four women and two men. Therefore the answer is $C(15,6) + C(15,5)C(10,1) + C(15,4)C(10,2) = 5005 + 30030 + 61425 = 96{,}460$.

36. Glue two 1's to the right of each 0, giving us a collection of nine tokens: five 011's and four 1's. We are asked for the number of strings consisting of these tokens. All that is involved is choosing the positions for the 1's among the nine positions in the string, so the answer is $C(9,4) = 126$.

38. $C(45,3) \cdot C(57,4) \cdot C(69,5) = 14190 \cdot 395010 \cdot 11238513 \approx 6.3 \times 10^{16}$

40. By the reasoning given in the solution to Exercise 41, the answer is $5!/(3 \cdot (5-3))! = 20$.

42. The only difference between this problem and the problem solved in Exercise 41 is a factor of 2. Each seating under the rules here corresponds to two seatings under the original rules, because we can change the order of people around the table from clockwise to counterclockwise. Therefore we need to divide the formula there by 2, giving us $n!/(2r(n-r)!)$. This assumes that $r \geq 3$. If $r = 1$ then the problem is trivial (there are n choices under both sets of rules). If $r = 2$, then we do not introduce the extra factor of 2, because clockwise order and counterclockwise order are the same. In this case, both answers are just $n!/(2(n-2)!)$, which is $C(n,2)$, as one would expect.

44. We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4,4) = 24$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4,2) = 6$ ways to choose the horses to be tied; then there are $P(3,3) = 6$ ways to determine the order of finish for the three groups (the pair and the two single horses). Thus there are $6 \cdot 6 = 36$ ways for this to happen. (3) There might be two groups of two horses that are tied. There are $C(4,2) = 6$ ways to choose the winners (and the other two horses are the losers). (4) There might be a group of three horses all tied. There are $C(4,3) = 4$

ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are $4 \cdot 2 = 8$ possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

46. **a)** The complicating factor here is the rule that the penalty kick round (or "group") is over once one team has clinched a victory. For example, if the first team to shoot has missed all of its first four shots and the other team has made two of its first three shots, then the round is over after only seven kicks. There are $2^{10} = 1024$ possible scenarios without this rule (and without worrying yet about whether the score is tied at the end of this round), but it seems rather tedious and dangerous (in the sense of your being likely to make a mistake and leave something out) to try to analyze the more complicated situation by writing out all the possibilities by hand. (This is not impossible, though, and the author has obtained the correct answer in this way.) Rather than do this, one can write a computer program to simulate the situation and do the counting. The result is that there are 672 possible scoring scenarios for a round of penalty kicks, including the possibility that the score is still tied at the end of that round.

Next we need to count the number of ways for the score to end up tied at the end of the round. For this to happen, both teams must score p points, where p is some integer between 0 and 5, inclusive. The scoring scenario is determined by the positions of the kickers who did the scoring. There are $C(5, p)$ ways to choose these positions for each team, or $C(5, p)^2$ ways in all. We need to sum this over the values of p from 0 to 5. The sum is 252. So there are 252 ways for the score to end up tied. We already noted in the paragraph above that there are 672 different scoring scenarios, so there are $672 - 252 = 420$ scenarios in which the score is not tied. This answers the question for this part of the exercise.

b) This is easy after what we've found above. There are 252 ways for the score to be tied at the end of the first group of penalty kicks, and there are 420 ways for the game to be settled in the second group. So there are $252 \cdot 420 = 105{,}840$ ways for the game to end during the second round.

c) We have already seen that there are 420 ways for the game to end in the first round, and 105,840 more ways for it to end in the second round. In order for it to go into a sudden death period, the first two rounds must have ended tied, which can happen in $420 \cdot 420 = 176{,}400$ ways. Thereafter, the game can end after two more kicks in 2 ways (either team can make their kick and have the other team miss theirs), after four more kicks in $2 \cdot 2 = 4$ ways (the first pair of kicks must have the same result, either both made or both missed, and then either team can win), after six more kicks in $2^2 \cdot 2 = 8$ ways (the first two pairs of kicks must have the same results, and then either team can win), after eight more kicks in 16 ways, and after ten more kicks in 32 ways. Thus there are $2 + 4 + 8 + 16 + 32 = 62$ ways for the sudden death round to end within ten kicks. This needs to be multiplied by the 176,400 ways we can reach sudden death, for a total of 10,936,800 scoring scenarios. So the answer to this last question is $420 + 105840 + 10936800 = 11{,}043{,}060$.

SECTION 6.4 Binomial Coefficients

2. **a)** When $(x + y)^5 = (x + y)(x + y)(x + y)(x + y)(x + y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, a term in the fourth sum, and a term in the fifth sum are added. Terms of the form x^5, x^4y, x^3y^2, x^2y^3, xy^4 and y^5 arise. To obtain a term of the form x^5, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^5 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{5}{5}$.) To obtain a term of the form x^4y, an x must be chosen in four of the five sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 4-combinations of five objects, namely $\binom{5}{4} = 5$. Similarly, the number of terms of the form x^3y^2 is the number of ways to pick three of the five sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{5}{3} = 10$ ways. By the same reasoning there are $\binom{5}{2} = 10$ ways to obtain the x^2y^3 terms, $\binom{5}{1}=5$ ways to obtain the xy^4 terms, and only one way (which we can think of as $\binom{5}{0}$) to obtain a y^5 term. Consequently, the product is $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

b) This is explained in Example 2. The expansion is $\binom{5}{0}x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+\binom{5}{5}y^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$. Note that it does not matter whether we think of the bottom of the binomial coefficient expression as corresponding to the exponent on x, as we did in part **(a)**, or the exponent on y, as we do here.

4. $\binom{13}{8}=1287$

6. $\binom{11}{7}1^4=330$

8. $\binom{17}{9}3^8 2^9 = 24310\cdot 6561\cdot 512 = 81{,}662{,}929{,}920$

10. By the binomial theorem, the typical term in this expansion is $\binom{100}{j}x^{100-j}(1/x)^j$, which can be rewritten as $\binom{100}{j}x^{100-2j}$. As j runs from 0 to 100, the exponent runs from 100 down to -100 in decrements of 2. If we let k denote the exponent, then solving $k=100-2j$ for j we obtain $j=(100-k)/2$. Thus the values of k for which x^k appears in this expansion are $-100,\ -98,\ \ldots,\ -2,\ 0,\ 2,\ 4,\ \ldots,\ 100$, and for such values of k the coefficient is $\binom{100}{(100-k)/2}$.

12. We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course):

$$1\quad 11\quad 55\quad 165\quad 330\quad 462\quad 462\quad 330\quad 165\quad 55\quad 11\quad 1$$

14. Using the factorial formulae for computing binomial coefficients, we see that $\binom{n}{k-1}=\frac{k}{n-k+1}\binom{n}{k}$. If $k\le n/2$, then $\frac{k}{n-k+1}<1$, so the "less than" signs are correct. Similarly, if $k>n/2$, then $\frac{k}{n-k+1}>1$, so the "greater than" signs are correct. The middle equality is Corollary 2 in Section 6.3, since $\lfloor n/2\rfloor+\lceil n/2\rceil=n$. The equalities at the ends are clear.

16. a) By Exercise 14, we know that $\binom{n}{\lfloor n/2\rfloor}$ is the largest of the $n-1$ binomial coefficients $\binom{n}{1}$ through $\binom{n}{n-1}$. Therefore it is at least as large as their average, which is $(2^n-2)/(n-1)$. But since $2n\le 2^n$ for $n\ge 2$, it follows that $(2^n-2)/(n-1)\ge 2^n/n$, and the proof is complete.

b) This follows from part **(a)** by replacing n with $2n$ when $n\ge 2$, and it is immediate when $n=1$.

18. The numeral 11 in base b represents the number $b+1$. Therefore the fourth power of this number is $b^4+4b^3+6b^2+4b+1$, where the binomial coefficients can be read from Pascal's triangle. As long as $b\ge 7$, these coefficients are single digit numbers in base b, so this is the meaning of the numeral $(14641)_b$. In short, the numeral formed by concatenating the symbols in the fourth row of Pascal's triangle is the answer.

20. It is easy to see that both sides equal

$$\frac{(n-1)!n!(n+1)!}{(k-1)!k!(k+1)!(n-k-1)!(n-k)!(n-k+1)!}\,.$$

22. a) Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and another, disjoint, subset with $r-k$ elements. The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of *these* elements are to go into the first of the subsets. The right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

b) On the one hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}\,,$$

and on the other hand

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}\,.$$

24. We know that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}\,.$$

Clearly p divides the numerator. On the other hand, p cannot divide the denominator, since the prime factorizations of these factorials contains only numbers less than p. Therefore the factor p does not cancel when this fraction is reduced to lowest terms (i.e., to a whole number), so p divides $\binom{p}{k}$.

26. First, use Exercise 25 to rewrite the right-hand side of this identity as $\binom{2n}{n+1}$. We give a combinatorial proof, showing that both sides count the number of ways to choose from collection of n men and n women, a subset that has one more man than woman. For the left-hand side, we note that this subset must have k men and $k-1$ women for some k between 1 and n, inclusive. For the (modified) right-hand side, choose any set of $n+1$ people from this collection of n men and n women; the desired subset is the set of men chosen and the women left behind.

28. **a)** To choose 2 people from a set of n men and n women, we can either choose 2 men ($\binom{n}{2}$ ways to do so) or 2 women ($\binom{n}{2}$ ways to do so) or one of each sex ($n\cdot n$ ways to do so). Therefore the right-hand side counts the number of ways to do this (by the sum rule). The left-hand side counts the same thing, since we are simply choosing 2 people from $2n$ people.

b) $2\binom{n}{2} + n^2 = n(n-1) + n^2 = 2n^2 - n = n(2n-1) = 2n(2n-1)/2 = \binom{2n}{2}$

30. We follow the hint. The number of ways to choose this committee is the number of ways to choose the chairman from among the n mathematicians (n ways) times the number of ways to choose the other $n-1$ members of the committee from among the other $2n-1$ professors. This gives us $n\binom{2n-1}{n-1}$, the expression on the right-hand side. On the other hand, for each k from 1 to n, we can have our committee consist of k mathematicians and $n-k$ computer scientists. There are $\binom{n}{k}$ ways to choose the mathematicians, k ways to choose the chairman from among these, and $\binom{n}{n-k}$ ways to choose the computer scientists. Since this last quantity equals $\binom{n}{k}$, we obtain the expression on the left-hand side of the identity.

32. For $n = 0$ we want

$$(x+y)^0 = \sum_{j=0}^{0}\binom{0}{j}x^{0-j}y^j = \binom{0}{0}x^0y^0\,,$$

which is true, since $1 = 1$. Assume the inductive hypothesis. Then we have

$$\begin{aligned}(x+y)^{n+1} &= (x+y)\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j\\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n+1-j}y^j + \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^{j+1}\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^k$$
$$= \binom{n}{0} x^{n+1} + \left(\sum_{k=1}^{n} [\binom{n}{k} + \binom{n}{k-1}] x^{n+1-k} y^k \right) + \binom{n}{n} y^{n+1}$$
$$= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + y^{n+1}$$
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k ,$$

as desired. The key point was the use of Pascal's identity to simplify the expression in brackets in the fourth line of this calculation.

34. By Exercise 33 there are $\binom{n-k+k}{k} = \binom{n}{k}$ paths from $(0,0)$ to $(n-k,k)$ and $\binom{k+n-k}{n-k} = \binom{n}{n-k}$ paths from $(0,0)$ to $(k, n-k)$. By symmetry, these two quantities must be the same (flip the picture around the 45° line).

36. A path ending up at $(n+1-k, k)$ must have made its last step either upward or to the right. If the last step was made upward, then it came from $(n+1-k, k-1)$; if it was made to the right, then it came from $(n-k, k)$. The path cannot have passed through both of these points. Therefore the number of paths to $(n+1-k, k)$ is the sum of the number of paths to $(n+1-k, k-1)$ and the number of paths to $(n-k, k)$. By Exercise 33 this tells us that $\binom{n+1-k+k}{k} = \binom{n+1-k+k-1}{k-1} + \binom{n-k+k}{k}$, which simplifies to $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, Pascal's identity.

38. We follow the hint, first noting that we can start the summation with $k = 1$, since the term with $k = 0$ is 0. The left-hand side counts the number of ways to choose a subset as described in the hint by breaking it down by the number of elements in the subset; note that there are k ways to choose each of the distinguished elements if the subset has size k. For the right-hand side, first note that $n(n+1)2^{n-2} = n(n-1+2)2^{n-2} = n(n-1)2^{n-2} + n2^{n-1}$. The first term counts the number of ways to make this choice if the two distinguished elements are different (choose them, then choose any subset of the remaining elements to be the rest of the subset). The second term counts the number of ways to make this choice if the two distinguished elements are the same (choose it, then choose any subset of the remaining elements to be the rest of the subset). Note that this works even if $n = 1$.

SECTION 6.5 Generalized Permutations and Combinations

2. There are 5 choices each of 5 times, so the answer is $5^5 = 3125$.

4. There are 6 choices each of 7 times, so the answer is $6^7 = 279{,}936$.

6. By Theorem 2 the answer is $C(3+5-1, 5) = C(7,5) = C(7,2) = 21$.

8. By Theorem 2 the answer is $C(21+12-1, 12) = C(32,12) = 225{,}792{,}840$.

10. **a)** $C(6+12-1,12) = C(17,12) = 6188$ **b)** $C(6+36-1,36) = C(41,36) = 749{,}398$

c) If we first pick the two of each kind, then we have picked $2 \cdot 6 = 12$ croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part **(a)**, namely $C(6+12-1,12) = C(17,12) = 6188$.

d) We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1,21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1,24) = C(29,24) = 118755$ ways to pick 24 croissants without any restriction, there must be $118755 - 65780 = 52{,}975$ ways to choose two dozen croissants with no more than two broccoli.

e) Eight croissants are specified, so this problem is the same as choosing $24 - 8 = 16$ croissants without restriction, which can be done in $C(6+16-1,16) = C(21,16) = 20{,}349$ ways.

f) First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are $24 - 9 = 15$ left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1,15) = C(20,15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1,11) = C(16,11) = 4368$ choices. Therefore the number of ways to make the selection without violating the restriction is $15504 - 4368 - 11{,}136$.

12. There are 5 things to choose from, repetitions allowed, and we want to choose 20 things, order not important. Therefore by Theorem 2 the answer is $C(5+20-1,20) = C(24,20) = C(24,4) = 10{,}626$.

14. By Theorem 2 the answer is $C(4+17-1,17) = C(20,17) = C(20,3) = 1140$.

16. **a)** We require each $x_i \geq 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = 17$ with each x_i' a nonnegative integer. The number of solutions is therefore $C(6+17-1,17) = C(22,17) = 26{,}334$.

b) The restrictions use up 22 of the total, leaving a free total of 7. Therefore the answer is $C(6+7-1,7) = C(12,7) = 792$.

c) The number of solutions without restriction is $C(6+29-1,29) = C(34,29) = 278256$. The number of solution violating the restriction by having $x_1 \geq 6$ is $C(6+23-1,23) = C(28,23) = 98280$. Therefore the answer is $278256 - 98280 = 179{,}976$.

d) The number of solutions with $x_2 \geq 9$ (as required) but without the restriction on x_1 is $C(6+20-1,20) = C(25,20) = 53130$. The number of solution violating the additional restriction by having $x_1 \geq 8$ is $C(6+12-1,12) = C(17,12) = 6188$. Therefore the answer is $53130 - 6188 = 46{,}942$.

18. It follows directly from Theorem 3 that the answer is

$$\frac{20!}{2!4!3!1!2!3!2!3!} \approx 5.9 \times 10^{13}.$$

20. We introduce the nonnegative slack variable x_4, and our problem becomes the same as the problem of counting the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 11$. By Theorem 2 the answer is $C(4+11-1,11) = C(14,11) = C(14,3) = 364$.

22. If we think of the balls as doing the choosing, then this is asking for the number of ways to choose 12 bins from the six given bins, with repetition allowed. (The number of times each bin is chosen is the number of balls in that bin.) By Theorem 2 with $n = 6$ and $r = 12$, this choice can be made in $C(6+12-1,12) = C(17,12) = 6188$ ways.

24. We assume that this problem leaves us free to pick which boxes get which numbers of balls. There are several ways to count this. Here is one. Line up the 15 objects in a row ($15!$ ways to do that), and line up the five boxes in a row ($5!$ ways to do that). Now put the first object into the first box, the next two into the second box, the next three into the third box, and so on. This overcounts by a factor of $1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!$, since there are that many ways to swap objects in the permutation without affecting the result. Therefore the answer is $15! \cdot 5!/(1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!) = 4{,}540{,}536{,}000$.

26. We can model this problem by letting x_i be the i^{th} digit of the number for $i = 1, 2, 3, 4, 5, 6$, and asking for the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$, where each x_i is between 0 and 8, inclusive, except that one of them equals 9. First, there are 6 ways to decide which of the digits is 9. Without loss of generality assume that $x_6 = 9$. Then the number of ways to choose the remaining digits is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$ (note that the restriction that each $x_i \le 8$ was moot, since the sum was only 4). By Theorem 2 there are $C(5 + 4 - 1, 4) = C(8, 4) = 70$ solutions. Therefore the answer is $6 \cdot 70 = 420$.

28. (Note that the roles of the letters n and r here are reversed from the usual roles, as, for example, in Theorem 2.) We can choose the required objects first, and there are $q_1 + q_2 + \cdots + q_r$ of these. Then $n - (q_1 + q_2 + \cdots + q_r) = n - q_1 - q_2 - \cdots - q_r$ objects remain to be chosen. There are still r types. Therefore by Theorem 2, the number of ways to make this choice is $C(r + (n - q_1 - q_2 - \cdots - q_r) - 1, (n - q_1 - q_2 - \cdots - q_r)) = C(n + r - q_1 - q_2 - \cdots - q_r - 1, n - q_1 - q_2 - \cdots - q_r)$.

30. By Theorem 3 the answer is $11!/(4!4!2!) = 34{,}650$.

32. We can treat the 3 consecutive A's as one letter. Thus we have 6 letters, of which 2 are the same (the two R's), so by Theorem 3 the answer is $6!/2! = 360$.

34. We need to calculate separately, using Theorem 3, the number of strings of length 5, 6, and 7. There are $7!/(3!3!1!) = 140$ strings of length 7. For strings of length 6, we can omit the R and form $6!/(3!3!) = 20$ string; omit an E and form $6!/(3!2!1!) = 60$ strings, or omit an S and also form 60 strings. This gives a total of 140 strings of length 6. For strings of length 5, we can omit two E's or two S's, each giving $5!/(3!1!1!) = 20$ strings; we can omit one E and one S ($5!/(2!2!1!) = 30$ strings); or we can omit the R and either an E or an S ($5!/(3!2!) = 10$ strings each). This gives a total of 90 strings of length 5, for a grand total of 370 strings of length 5 or greater.

36. We simply need to choose the 6 positions, out of the 14 available, to make 1's. There are $C(14, 6) = 3003$ ways to do so.

38. We assume that the forty issues are distinguishable.

a) Theorem 4 says that the answer is $40!/10!^4 \approx 4.7 \times 10^{21}$.

b) Each distribution into identical boxes gives rise to $4! = 24$ distributions into labeled boxes, since once we have made the distribution into unlabeled boxes we can arbitrarily label the boxes. Therefore the answer is the same as the answer in part **(a)** divided by 24, namely $(40!/10!^4)/4! \approx 2.0 \times 10^{20}$.

40. We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, 5 z's, and 4 w's. By Theorem 3, there are

$$\frac{16!}{4!3!5!4!} = 50{,}450{,}400$$

such sequences.

42. Theorem 4 says that the answer is $52!/13!^4 \approx 5.4 \times 10^{28}$, since each player gets 13 cards.

44. **a)** All that matters is the number of books on each shelf, so the answer is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$, where x_i is being viewed as the number of books on shelf i. The answer is therefore $C(4 + 12 - 1, 12) = C(15, 12) = 455$.
b) No generality is lost if we number the books b_1, b_2, ..., b_{12} and think of placing book b_1, then placing b_2, and so on. There are clearly 4 ways to place b_1, since we can put it as the first book (for now) on any of the shelves. After b_1 is placed, there are 5 ways to place b_2, since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are 6 ways to place b_3 (to the right of b_1, to the right of b_2, or as the first book on any of the shelves), 7 ways to place b_4, ..., 15 ways to place b_{12}. Therefore the answer is the product of these numbers $4 \cdot 5 \cdots 15 = 217{,}945{,}728{,}000$.

46. We follow the hint. There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars, ..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore $C(6 + 3 - 1, 3) = C(8, 3) = 56$.

48. We can think of the n distinguishable objects to be distributed into boxes as numbered from 1 to n. Since such a distribution is completely determined by assigning a box number (from 1 to k) to each object, we can think of a distribution simply as a sequence of box numbers a_1, a_2, ..., a_n, where a_i is the box into which object i goes. Furthermore, since we want n_i objects to go into box i, this sequence must contain n_i copies of the number i (for each i from 1 to k). But this is precisely a permutation of n objects (namely, numbers) with n_i indistinguishable objects of type i (namely, n_i copies of the number i). Thus we have established the desired one-to-one correspondence. Since Theorem 3 tells us that there are $n!/(n_1!n_2! \cdots n_k!)$ permutations, there must also be this many ways to do the distribution into boxes, and the proof of Theorem 4 is complete.

50. This is actually a problem about partitions of sets. Let us call the set of 5 objects $\{a, b, c, d, e\}$. We want to partition this set into three pairwise disjoint subsets (some possibly empty). We count in a fairly ad hoc way. First, we could put all five objects into one subset (i.e., all five objects go into one box, with the other two boxes empty). Second, we could put four of the objects into one subset and one into another, such as $\{a, b, c, d\}$ together with $\{e\}$. There are 5 ways to do this, since each of the five objects can be the singleton. Third, we could put three of the objects into one set (box) and two into another; there are $C(5, 2) = 10$ ways to do this, since there are that many ways to choose which objects are to be the doubleton. Similarly, there are 10 ways to distribute the elements so that three go into one set and one each into the other two sets (for example, $\{a, b, c\}$, $\{d\}$, and $\{e\}$). Finally, we could put two items into one set, two into another, and one into the third (for example, $\{a, b\}$, $\{c, d\}$, and $\{e\}$). Here we need to choose the singleton (5 ways), and then we need to choose one of the 3 ways to separate the remaining four elements into pairs; this gives a total of 15 partitions. In all we have 41 different partitions.

This can also be solved by using the formulae given in the text in a discussion of Stirling numbers of the second kind (this follows Example 10):

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32 - 2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243 - 96 + 3) = 25$$

$$\sum_{j=1}^{3} S(5,j) = 1 + 15 + 25 = 41$$

52. This is similar to Exercise 50, with 3 replaced by 4. We compute this using the formulae:

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32-2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243-96+3) = 25$$

$$S(5,4) = \frac{1}{4!}\left(\binom{4}{0}4^5 - \binom{4}{1}3^5 + \binom{4}{2}2^5 - \binom{4}{3}1^5\right) = \frac{1}{4!}(1024-972+192-4) = 10$$

$$\sum_{j=1}^{4} S(5,j) = 1 + 15 + 25 + 10 = 51$$

54. We are asked for the partitions of 5 into at most 3 parts; notice that we are not required to use all three boxes. We can easily list these partitions explicitly: $5=5$, $5=4+1$, $5=3+2$, $5=3+1+1$, and $5=2+2+1$. Therefore the answer is 5.

56. This is similar to Exercise 55. Since each box has to contain at least one object, we might as well put one object into each box to begin with. This leaves us with just three more objects, and there are only three choices: we can put them all into the same box (so that the partition we end up with is $8=4+1+1+1+1$), or we can put them into three different boxes (so that the partition we end up with is $8=2+2+2+1+1$), or we can put two into one box and the last into another (so that the partition we end up with is $8=3+2+1+1+1$). So the answer is 3.

58. **a)** This is a straightforward application of the product rule: There are 7 choices for the first ball, 6 choices for the second ball, and so on, for an answer of $7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520$.

b) Since each ball must be in a separate box and the boxes are unlabeled, there is only one way to do this.

c) This is just a matter of choosing which five boxes to put balls into, so the answer is $C(7,5)=21$.

d) As noted in part **(b)**, there is only one way to do this.

60. There are 31 other teams to play, and we can denote these with the symbols x_1, x_2, ..., x_{31}. We are asked for a list of $4\cdot 4 + 11\cdot 3 + 16\cdot 2 = 81$ of these symbols that contains exactly 4 copies of each of x_1 through x_4, exactly 3 copies of each of x_5 through x_{15}, and exactly 2 copies of each of x_{16} through x_{31}. Theorem 3 tells us that the number of possible lists is

$$\frac{81!}{(4!)^4\cdot(3!)^{11}\cdot(2!)^{16}} \approx 7.35\times 10^{101}.$$

(The arithmetic was done with *Maple*.)

62. Each term must be of the form $Cx_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, where the n_i's are nonnegative integers whose sum is n. The number of ways to specify a term, then, is the number of nonnegative integer solutions to $n_1+n_2+\cdots+n_m=n$, which by Theorem 2 is $C(m+n-1,n)$. Note that the coefficients C for these terms can be computed using Theorem 3—see Exercise 63.

64. From Exercise 62, we know that there are $C(3+4-1,4) = C(6,4) = 15$ terms, and the coefficients come from Exercise 63. The answer is $x^4 + y^4 + z^4 + 4x^3y + 4xy^3 + 4x^3z + 4xz^3 + 4y^3z + 4yz^3 + 6x^2y^2 + 6x^2z^2 + 6y^2z^2 + 12x^2yz + 12xy^2z + 12xyz^2$.

66. By Exercise 62, the answer is $C(3+100-1,100) = C(102,100) = C(102,2) = 5151$.

SECTION 6.6 Generating Permutations and Combinations

2. 156423, 165432, 231456, 231465, 234561, 314562, 432561, 435612, 541236, 543216, 654312, 654321

4. Our list will have $3^3 \cdot 2^2 = 108$ items in it. Here it is in lexicographic order: 000aa, 000ab, 000ba, 000bb, 001aa, 001ab, 001ba, 001bb, 002aa, 002ab, 002ba, 002bb, 010aa, 010ab, 010ba, 010bb, 011aa, 011ab, 011ba, 011bb, 012aa, 012ab, 012ba, 012bb, 020aa, 020ab, 020ba, 020bb, 021aa, 021ab, 021ba, 021bb, 022aa, 022ab, 022ba, 022bb, 100aa, 100ab, 100ba, 100bb, 101aa, 101ab, 101ba, 101bb, 102aa, 102ab, 102ba, 102bb, 110aa, 110ab, 110ba, 110bb, 111aa, 111ab, 111ba, 111bb, 112aa, 112ab, 112ba, 112bb, 120aa, 120ab, 120ba, 120bb, 121aa, 121ab, 121ba, 121bb, 122aa, 122ab, 122ba, 122bb, 200aa, 200ab, 200ba, 200bb, 201aa, 201ab, 201ba, 201bb, 202aa, 202ab, 202ba, 202bb, 210aa, 210ab, 210ba, 210bb, 211aa, 211ab, 211ba, 211bb, 212aa, 212ab, 212ba, 212bb, 220aa, 220ab, 220ba, 220bb, 221aa, 221ab, 221ba, 221bb, 222aa, 222ab, 222ba, 222bb.

6. These can be done using Algorithm 1 or Example 2. This will be explained in detail for part **(a)**; the others are similar. In the last four parts of this exercise, the next permutation exchanges only the last two elements.
a) The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_2 = 3$ and $a_3 = 4$. The least integer to the right of 3 that is greater than 3 is 4. Hence 4 is placed in the second position. The integers 2 and 3 are then placed in order in the last two positions, giving the permutation 1423.
b) 51234 **c)** 13254 **d)** 612354 **e)** 1623574 **f)** 23587461

8. The first subset corresponds to the bit string 0000, namely the empty set. The next subset corresponds to the bit string 0001, namely the set $\{4\}$. The next bit string is 0010, corresponding to the set $\{3\}$, and then 0011, which corresponds to the set $\{3,4\}$. We continue in this manner, giving the remaining sets: $\{2\}$, $\{2,4\}$, $\{2,3\}$, $\{2,3,4\}$, $\{1\}$, $\{1,4\}$, $\{1,3\}$, $\{1,3,4\}$, $\{1,2\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{1,2,3,4\}$.

10. Since the new permutation agrees with the old one in positions 1 to $j-1$, and since the new permutation has a_k in position j, whereas the old one had a_j, with $a_k > a_j$, the new permutation succeeds the old one in lexicographic order. Furthermore the new permutation is the first one (in lexicographic order) with a_1, a_2, $\ldots$, a_{j-1}, a_k in positions 1 to j, and the old permutation was the last one with a_1, a_2, $\ldots$, a_{j-1}, a_j in those positions. Since a_k was picked to be the smallest number greater than a_j among a_{j+1}, a_{j+2}, $\ldots$, a_n, there can be no permutation between these two.

12. One algorithm would combine Algorithm 3 and Algorithm 1. Using Algorithm 3, we generate all the r-combinations of the set with n elements. At each stage, after we have found each r-combination, we use Algorithm 1, with $n = r$ (and a different collection to be permuted than $\{1, 2, \ldots, n\}$), to generate all the permutations of the elements in this combination. See the solution to Exercise 13 for an example.

14. **a)** We find that $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 2$, and $a_5 = 3$. Therefore the number is $1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 2 \cdot 4! + 3 \cdot 5! = 1 + 2 + 12 + 48 + 360 = 423$.
b) Each $a_k = 0$, so the number is 0.
c) We find that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, and $a_5 = 5$. Therefore the number is $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! = 1 + 4 + 18 + 96 + 600 = 719 = 6! - 1$, as expected, since this is the last permutation.

16. **a)** We find the Cantor expansion of 3 to be $1\cdot 1! + 1\cdot 2!$. Therefore we know that $a_4 = 0$, $a_3 = 0$, $a_2 = 1$, and $a_1 = 1$. Following the algorithm given in the solution to Exercise 15, we put 5 in position $5 - 0 = 5$, put 4 in position $4 - 0 = 4$, put 3 in position $3 - 1 = 2$, and put 2 in the position that is 1 from the rightmost available position, namely position 1. Therefore the answer is 23145.

b) We find that $89 = 1\cdot 1! + 2\cdot 2! + 2\cdot 3! + 3\cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 3, 2, 2, and 1 positions from the right among the available positions, obtaining 35421.

c) We find that $111 = 1\cdot 1! + 1\cdot 2! + 2\cdot 3! + 4\cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 4, 2, 1, and 1 positions from the right among the available positions, obtaining 52431.

SUPPLEMENTARY EXERCISES FOR CHAPTER 6

2. **a)** There are no ways to do this, since there are not enough items. **b)** $6^{10} = 60{,}466{,}176$

c) There are no ways to do this, since there are not enough items.

d) $C(6 + 10 - 1, 10) = C(15, 10) = C(15, 5) = 3003$

4. There are 2^7 bit strings of length 10 that start 000, since each of the last 7 bits can be chosen in either of two ways. Similarly, there are 2^6 bit strings of length 10 that end 1111, and there are 2^3 bit strings of length 10 that both start 000 and end 1111 (since only the 3 middle bits can be freely chosen). Therefore by the inclusion–exclusion principle, the answer is $2^7 + 2^6 - 2^3 = 184$.

6. $9\cdot 10\cdot 10\cdot 10\cdot 10 = 90{,}000$

8. **a)** All the integers from 100 to 999 have three decimal digits, and there are $999 - 100 + 1 = 900$ of these.

b) In addition to the 900 three-digit numbers, there are 9 one-digit positive integers, for a total of 909.

c) There is 1 one-digit number with a 9. Among the two-digit numbers, there are the 10 numbers from 90 to 99, together with the 8 numbers $19, 29, \ldots, 89$, for a total of 18. Among the three-digit numbers, there are the 100 from 900 to 999; and there are, for each century from the 100's to the 800's, again $1 + 18 = 19$ numbers with at least one 9; this gives a total of $100 + 8\cdot 19 = 252$. Thus our final answer is $1 + 18 + 252 = 271$. Alternately, we can compute this as $10^3 - 9^3 = 271$, since we want to subtract from the number of three-digit nonnegative numbers (with leading 0's allowed) the number of those that use only the nine digits 0 through 8.

d) Since we can use only even digits, there are $5^3 = 125$ ways to specify a three-digit number, allowing leading 0's. Since, however, the number $0 = 000$ is not in our set, we need to subtract 1, obtaining the answer 124.

e) The numbers in question are either of the form $d55$ or $55d$, with $d \neq 5$, or 555. Since d can be any of nine digits, there are $9 + 9 + 1 = 19$ such numbers.

f) All 9 one-digit numbers are palindromes. The 9 two-digit numbers $11, 22, \ldots, 99$ are palindromes. For three-digit numbers, the first digit (which must equal the third digit) can be any of the 9 nonzero digits, and the second digit can be any of the 10 digits, giving $9\cdot 10 = 90$ possibilities. Therefore the answer is $9 + 9 + 90 = 108$.

10. Using the generalized pigeonhole principle, we see that we need $5 \times 12 + 1 = 61$ people.

12. There are $7 \times 12 = 84$ day-month combinations. Therefore we need 85 people to ensure that two of them were born on the same day of the week and in the same month.

14. We need at least 551 cards to ensure that at least two are identical. Since the cards come in packages of 20, we need $\lceil 551/20 \rceil = 28$ packages.

16. Partition the set of numbers from 1 to $2n$ into the n pigeonholes $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{2n-1,2n\}$. If we have $n+1$ numbers from this set (the pigeons), then two of them must be in the same hole. This means that among our collection are two consecutive numbers. Clearly consecutive numbers are relatively prime (since every common divisor must divide their difference, 1).

18. Divide the interior of the square, with lines joining the midpoints of opposite sides, into four 1×1 squares. By the pigeonhole principle, at least two of the five points must be in the same small square. The furthest apart two points in a square could be is the length of the diagonal, which is $\sqrt{2}$ for a square 1 unit on a side.

20. If the worm never gets sent to the same computer twice, then it will infect 100 computers on the first round of forwarding, $100^2 = 10{,}000$ other computers on the second round of forwarding, and so on. Therefore the maximum number of different computers this one computer can infect is $100 + 100^2 + 100^3 + 100^4 + 100^5 = 10{,}101{,}010{,}100$. This figure of ten billion is probably comparable to the total number of computers in the world.

22. **a)** We want to solve $n(n-1) = 110$, or $n^2 - n - 110 = 0$. Simple algebra gives $n = 11$ (we ignore $n = -10$, since we need a positive integer for our answer).

b) We recall that $7! = 5040$, so the answer is 7.

c) We need to solve the equation $n(n-1)(n-2)(n-3) = 12n(n-1)$. Since we have $n \geq 4$ in order for $P(n,4)$ to be defined, this equation reduces to $(n-2)(n-3) = 12$, or $n^2 - 5n - 6 = 0$. Simple algebra gives $n = 6$ (we ignore the solution $n = -1$ since n needs to be a positive integer).

24. An algebraic proof is straightforward. We will give a combinatorial proof of the equivalent identity $P(n+1,r)(n+1-r) = (n+1)P(n,r)$ (and in fact both of these equal $P(n+1,r+1)$). Consider the problem of writing down a permutation of $r+1$ objects from a collection of $n+1$ objects. We can first write down a permutation of r of these objects ($P(n+1,r)$ ways to do so), and then write down one more object (and there are $n+1-r$ objects left to choose from), thereby obtaining the left-hand side; or we can first choose an object to write down first ($n+1$ to choose from), and then write down a permutation of length r using the n remaining objects ($P(n,r)$ ways to do so), thereby obtaining the right-hand side.

26. First note that Corollary 2 of Section 6.4 is equivalent to the assertion that the sum of the numbers $C(n,k)$ for even k is equal to the sum of the numbers $C(n,k)$ for odd k. Since $C(n,k)$ counts the number of subsets of size k of a set with n elements, we need to show that a set has as many even-sized subsets as it has odd-sized subsets. Define a function f from the set of all subsets of A to itself (where A is a set with n elements, one of which is a), by setting $f(B) = B \cup \{a\}$ if $a \notin B$, and $f(B) = B - \{a\}$ if $a \in B$. It is clear that f takes even-sized subsets to odd-sized subsets and vice versa, and that f is one-to-one and onto (indeed, $f^{-1} = f$). Therefore f restricted to the set of subsets of odd size gives a one-to-one correspondence between that set and the set of subsets of even size.

28. The base case is $n = 2$, in which case the identity simply states that $1 = 1$. Assume the inductive hypothesis, that $\sum_{j=2}^{n} C(j,2) = C(n+1,3)$. Then

$$\begin{aligned}\sum_{j=2}^{n+1} C(j,2) &= \left(\sum_{j=2}^{n} C(j,2)\right) + C(n+1,2)\\ &= C(n+1,3) + C(n+1,2) = C((n+1)+1,3)\,,\end{aligned}$$

as desired. The last equality made use of Pascal's identity.

30. Each pair of values of i and j with $1 \leq i < j \leq n$ contributes a 1 to this sum, so the sum is just the number of such pairs. But this is clearly the number of ways to choose two integers from $\{1,2,\ldots,n\}$, which is $C(n,2)$, also known as $\binom{n}{2}$.

32. **a)** For a fixed k, a triple is totally determined by picking i and j; since each can be picked in k ways (each can be any number from 0 to $k-1$, inclusive), there are k^2 ways to choose the triple. Adding over all possible values of k gives the indicated sum.

b) A triple of this sort is totally determined by knowing the *set* of numbers $\{i,j,k\}$, since the order is fixed. Therefore the number of triples of each kind is just the number of sets of 3 elements chosen from the set $\{0,1,2,\ldots,n\}$, and that is clearly $C(n+1,3)$.

c) In order for i to equal j (with both less than k), we need to pick two elements from $\{0,1,2,\ldots,n\}$, using the larger one for k and the smaller one for both i and j. Therefore there are as many such choices as there are 2-element subsets of this set, namely $C(n+1,2)$.

d) This part is its own proof. The last equality follows from elementary algebra.

34. **a)** If we 2-color the $2d-1$ elements of S, then there must be at least d elements of one color (if there were $d-1$ or fewer elements of both colors, then only $2d-2$ elements would be colored); this is just an application of the generalized pigeonhole principle. Thus there is a d-element subset that does not contain both colors, in violation of the condition for being 2-colorable.

b) We must show that every collection of fewer than three sets each containing two elements is 2-colorable, and that there is a collection of three sets each containing two elements that is not 2-colorable. The second statement follows from part **(a)**, with $d=2$ (the three sets are $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$). On the other hand, if we have two (or fewer) sets each with two elements, then we can color the two elements of the first set with different colors, and we cannot be prevented from properly coloring the second set, since it must contain an element not in the first set.

c) First we show that the given collection is not 2-colorable. Without loss of generality, assume that 1 is red. If 2 is red, then 6 must be blue (second set). Thus either 4 or 5 must be red (seventh set), which means that 3 must be blue (first or fourth set). This would force 7 to be red (sixth set), which would force both 4 and 5 to be blue (third and fifth sets), a contradiction. Thus 2 is blue. If 3 is red, then we can conclude that 5 is blue, 7 is red, 6 is blue, and 4 is blue, making the last set improperly colored. Thus 3 is blue. This implies that 4 is red, hence 7 is blue, hence 5 and 6 are red, another contradiction. So the given collection cannot be 2-colored. Next we must show that all collections of six sets with three elements each are 2-colorable. Since having more elements in S at our disposable only makes it easier to 2-color the collection, we can assume that S has only five elements; let $S=\{a,b,c,d,e\}$. Since there are 18 occurrences of elements in the collection, some element, say a, must occur at least four times (since $3\cdot 5<18$). If a occurs in six of the sets, then we can color a red and the rest of the elements blue. If a occurs in five of the sets, suppose without loss of generality that b and c occur in the sixth set. Then we can color a and b red and the remaining elements blue. Finally, if a occurs in only four of the sets, then that leaves only four elements for the last two sets, and therefore a pair of elements must be shared by them, say b and c. Again coloring a and b red and the remaining elements blue gives the desired coloring.

36. We might as well assume that the first person sits in the northernmost seat. Then there are $P(7,7)$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is $7!=5040$.

38. We need to know the number of solutions to $d+m+g=12$, where d, m, and g are integers greater than or equal to 3. This is equivalent to the number of nonnegative integer solutions to $d'+m'+g'=3$, where $d'=d-3$, $m'=m-3$, and $g'=g-3$. By Theorem 2 of Section 6.5, the answer is $C(3+3-1,3)=C(5,3)=10$.

40. **a)** By Theorem 3 of Section 6.5, the answer is $10!/(3!2!2!)=151{,}200$.

b) If we fix the start and the end, then the question concerns only 8 letters, and the answer is $8!/(2!2!)=10{,}080$.

c) If we think of the three P's as one letter, then the answer is seen to be $8!/(2!2!) = 10{,}080$.

42. There are 26 choices for the third letter. If the digit part of the plate consists of the digits 1, 2, and d, where d is different from 1 or 2, then there are 8 choices for d and $3! = 6$ choices for a permutation of these digits. If $d = 1$ or 2, then there are 2 choices for d and 3 choices for a permutation. Therefore the answer is $26(8 \cdot 6 + 2 \cdot 3) = 1404$.

44. Let us look at the girls first. There are $P(8,8) = 8! = 40320$ ways to order them relative to each other. This much work produces 9 gaps between girls (including the ends), in each of which at most one boy may sit. We need to choose, in order without repetition, 6 of these gaps, and this can be done in $P(9,6) = 60480$ ways. Therefore the answer is, by the product rule, $40320 \cdot 60480 = 2{,}438{,}553{,}600$.

46. We are given no restrictions, so any number of the boxes can be occupied once we have distributed the objects.
a) This is a straightforward application of the product rule; there are $6^5 = 7776$ ways to do this, because there are 6 choices for each of the 5 objects.
b) This is similar to Exercise 50 in Section 6.5. We compute this using the formulae:

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32-2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243-96+3) = 25$$

$$S(5,4) = \frac{1}{4!}\left(\binom{4}{0}4^5 - \binom{4}{1}3^5 + \binom{4}{2}2^5 - \binom{4}{3}1^5\right) = \frac{1}{4!}(1024-972+192-4) = 10$$

$$S(5,5) = \frac{1}{5!}\left(\binom{5}{0}5^5 - \binom{5}{1}4^5 + \binom{5}{2}3^5 - \binom{5}{3}2^5 + \binom{5}{4}1^5\right) = \frac{1}{5!}(3125-5120+2430-320+5) = 1$$

$$\sum_{j=1}^{5} S(5,j) = 1 + 15 + 25 + 10 + 1 = 52$$

c) This is asking for the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5$ in nonnegative integers. By Theorem 2 (see also Example 5) in Section 6.5, the answer is $C(6+5-1,5) = C(10,5) = 252$.
d) This is asking for the number of partitions of 5 (into at most six parts, but that is moot). We list them: $5 = 5$, $5 = 4+1$, $5 = 3+2$, $5 = 3+1+1$, $5 = 2+2+1$, $5 = 2+1+1+1$, $5 = 1+1+1+1+1$. Therefore the answer is 7.

48. One way to look at this involves what is called the cycle structure of a permutation. Think of the people as the numbers from 1 to n. Given a permutation π of $\{1, 2, \ldots, n\}$, we can write down the cycles the result from applying this permutation. Each cycle can be viewed as a list of the people sitting at a circular table, in clockwise order. The first cycle contains 1, $\pi(1)$, $\pi(\pi(1))$, $\ldots$, until we eventually return to 1 (which must happen because permutation are one-to-one functions). If k is the first number not in the first cycle, then the second cycle consists of k, $\pi(k)$, $\pi(\pi(k))$, $\ldots$, and so on. For example, the permutation that sends x to $x+3$ on a 12-hour clock has cycle structure $(1,4,7,10)$, $(2,5,8,11)$, $(3,6,9,12)$. Thus each of the $n!$ permutations gives rise to a seating of n people around j circular tables for some j between 1 and n inclusive. Conversely, such a seating gives us a permutation—$\pi(x)$ is the number clockwise from x at whatever table x is at (which could be x itself). The identity follows from this discussion.

50. We can give a nice combinatorial proof here. If we wish to have people numbered 1 through $n+1$ sit at k circular tables, there are two choices. We could have $n+1$ sit at a table by himself and then place the remaining n people at $k-1$ circular tables (the first term on the right-hand side of this identity), or we could seat the first n people at the k tables and then have $n+1$ sit immediately to the right of one of those people (there being n choices for this last step, giving us the second term on the right).

52. Except for the last three symbols, for which we have no choice, we need a permutation of 2 A's, 2 C's, 2 U's, and 2 G's. By Theorem 3 in Section 6.5, the answer is $8!/(2!)^4 = 2520$.

54. From the first piece of information, we know that the chain ends AC and preceding that are the chains UG and ACG in some order. So there are only two choices: UGACGAC or ACGUGAC. It is easily seen that breaking the first of these after each U or C produces the fragments stated in the second half of the first sentence, whereas breaking the second choice similarly produces something else (AC, GU, GAC). Therefore the original chain was UGACGAC.

56. Assume without loss of generality that we wish to form r-combinations from the set $\{1, 2, \ldots, n\}$. We modify Algorithm 3 in Section 6.6 for generating the next r-combination in lexicographic order, allowing for repetition. Then we generate all such combinations by starting with $11\ldots1$ and calling this modified algorithm $C(n+r-1, r) - 1$ times (this will give us $nn\ldots n$ as the last one).

procedure *next r-combination*$(a_1, a_2, \ldots, a_r$: integers)
$\{$We assume that $1 \le a_1 \le a_2 \le \cdots \le a_r \le n$, with $a_1 \ne n\,\}$
$i := r$
while $a_i = n$
    $i := i - 1$
$a_i := a_i + 1$
for $j := i + 1$ **to** r
    $a_j := a_i$

58. One needs to play around with this enough to eventually discover a situation satisfying the conditions. Here is a way to do it. Suppose the group consists of three men and three women, and suppose that people of the same sex are always enemies and people of the opposite sex are always friends. Then clearly there can be no set of four mutual enemies, because any set of four people must include at least one man and one woman (since there are only three of each sex in the whole group). Also there can be no set of three mutual friends, because any set of three people must include at least two people of the same sex (since there are only two sexes).